Vector fields on the plane

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Vector fields

Definition

Suppose at each point of the plane $\mathbb{R}^2$ there is given a vector, so that the coordinates of the vector vary continuously with the point. Then we say we are given a **vector field** on $\mathbb{R}^2$.

Often convenient to give vector field in the form $f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$. This means that at any point (x,y) we are given the vector with coordinates $(f(x,y), g(x,y))$.

Definition

Points with the zero vector assigned are called **singular**. We will always assume that our vector fields have finitely many singular points.

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- When x goes along C, vector $\tilde{v}_x$ will rotate. Count the total number of rotations, clock-wise ones counted with "$-$" sign and counter-clock-wise with the "$+$" sign.
- This **total number** of rotations is called the **index** of C, denoted $i(C)$.

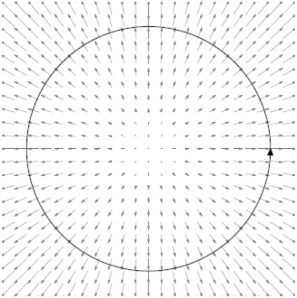
Index of a singular point

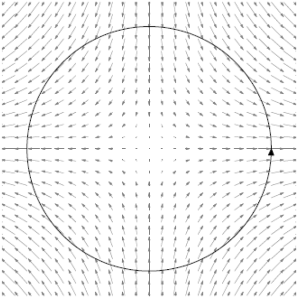
Definition

Suppose $z \in \mathbb{R}^2$ is a singular point. Take a closed curve C around z which does not contain any other singular points. Then $i(C)$ is called **index of** z and is denoted by $i(z)$.

Notice: this definition makes sense!

Exercise: compute indices of singular points of the fields below.





Trajectories

We can think of a vector field as defining velocities of each point. So every point is moving along a trajectory, and so the vector field is a field of velocities of the points moving along the trajectories.

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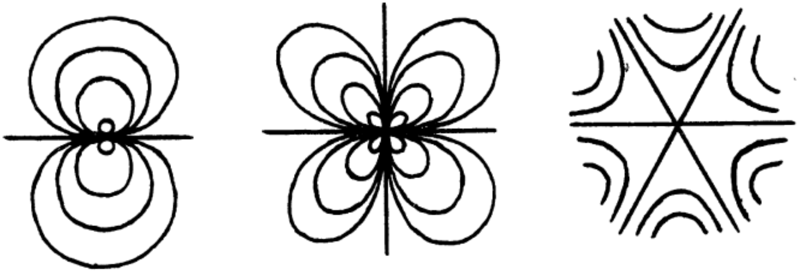

Index theorem

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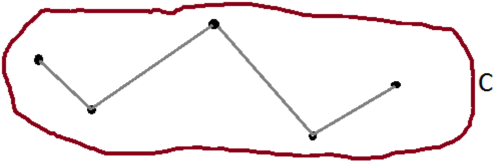

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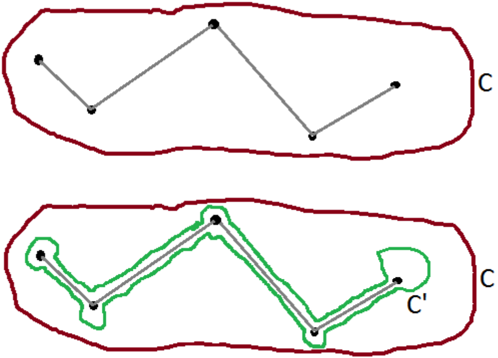

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Application:

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Let $f : D \to D$ *be a continuous map from a disk to itself, such that each point of* $S^1 = \partial D$ *is mapped to itself. Then there exists a point* $x \in D$ *mapping to the center* O *of* D*.*

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This is what we wanted.

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Any polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *with complex coefficients has a complex root.*

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When φ goes from 0 to 2π, $n\varphi$ goes from 0 to $2\pi n$. Done.

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So w has a singular point inside C, i.e. the polynomial $P(z)$ has a root inside C.